TRIM-NP Documentation Manual

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Chapter 2

Theory

2.1 Governing Equations

This section presents a brief summary of the theory behind the TRIMNP code. It follows closely the two key publications [1] and [2]. For more details the reader is referred to these publications.

2.1.1 Navier-Stokes Equations

Starting point for a general discussion are the three-dimensional non-hydrostatic Navier-Stokes equations after Reynolds averaging:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right) + fv \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right) - fu \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial w}{\partial z}\right) - g \end{aligned} \tag{2.1.1}$$

Splitting the pressure p in equation 2.1.1 into a hydrostatic component p_H and into a non-hydrostatic component q results in:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p_H}{\partial x} - \frac{\partial q}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right) + fv \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p_H}{\partial y} - \frac{\partial q}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right) - fu \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{\partial q}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial w}{\partial z}\right) \end{aligned} \tag{2.1.2}$$

Note that the scalar q has the dimension of p/ρ. If the model is run in the hydrostatic mode all q-terms in equation 2.1.2 are simply neglected. The horizontal viscosity μ is a constant specified in the input data set. The vertical viscosity ν is determined by the kind of turbulence parameterization implemented.

2.1.2 Barotropic and Baroclinic Pressure Gradient

The hydrostatic pressure gradient can be subdivided into barotropic and baroclinic parts, respectively. From the hydrostatic relation

$$\frac{\partial p_H}{\partial z} = -\rho g \tag{2.1.3}$$

the pressure at level z is found by integrating from the surface η to z:

$$p_H(z) = p(\eta) + g \int_z^\eta \rho dz \tag{2.1.4}$$

Here $p(\eta)$ corresponds to the atmospheric pressure at the free surface. If the density is expressed as $\rho = \rho_0 + \rho'$ the following relation results from 2.1.4:

$$p_H(z) = p(\eta) + g\rho_0(\eta - z) + g \int_z^\eta \rho' dz \tag{2.1.5}$$

We are interested in horizontal gradients only. Therefore, taking the derivative of 2.1.5 with respect to x yields

$$\frac{\partial p_H}{\partial x} = g\rho_0 \frac{\partial \eta}{\partial x} + g\frac{\partial}{\partial x} \int_z^{\eta(x)} \rho' dz \tag{2.1.6}$$

where the atmospheric pressure gradient has been neglected. It must be reconsidered for extreme events like hurricans. The term proportial to the gradient of η is the *barotropic* pressure gradient, while the last term is called the *baroclinic* pressure gradient. The latter can be transformed further by applying the Leibniz rule:

$$\frac{\partial}{\partial x} \int_z^{\eta(x)} \rho' dz = \rho'(\eta)\frac{\partial \eta}{\partial x} - \rho'(z)\frac{\partial z}{\partial x} + \int_z^\eta \frac{\partial \rho'}{\partial x} dz \tag{2.1.7}$$

The first term on the right-hand-side of 2.1.7 can be added to the barotropic pressure gradient. The second term is zero since z does not depend on x. As a result the pressure gradient terms of the Navier-Stokes equations 2.1.2 can now be written as:

$$\begin{aligned} -\frac{1}{\rho}\frac{\partial p_H}{\partial x} &= -g\frac{\partial \eta}{\partial x} - \frac{g}{\rho_0}\int_z^\eta \frac{\partial \rho'}{\partial x} dz \\ -\frac{1}{\rho}\frac{\partial p_H}{\partial y} &= -g\frac{\partial \eta}{\partial y} - \frac{g}{\rho_0}\int_z^\eta \frac{\partial \rho'}{\partial y} dz \end{aligned} \tag{2.1.8}$$

where $1/\rho$ of the baroclinic term has been approximated by $1/\rho_0$.

2.1.3 Continuity Equation

Conservation of mass is expressed by the continuity equation which has been simplified to an expression for incompressibility:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.1.9}$$

By integrating equation 2.1.9 vertically from the bottom $z = -h$ to the free surface η a prognostic equation for the free surface results:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[\int_{-h}^\eta u\ dz\right] + \frac{\partial}{\partial y}\left[\int_{-h}^\eta v\ dz\right] = 0 \tag{2.1.10}$$

2.1.4 Boundary Conditions

The upper boundary condition at the free surface is a wind stress formulated as

$$\nu \frac{\partial u}{\partial z} = \tau_x^w \quad \text{and} \quad \nu \frac{\partial v}{\partial z} = \tau_y^w \tag{2.1.11}$$

The wind stress is expressed as a function of the wind speed given at 10 m height U_{10}^w:

$$\tau^w = c_D \cdot \frac{\rho_a}{\rho} \cdot (U_{10}^w - u) \cdot |U_{10}^w| \tag{2.1.12}$$

with the drag coefficient c_D determined from experiments. The ratio of atmospheric density ρ_a to water density ρ is set to a constant value of $1.25 \cdot 10^{-3}$. For c_D a parameterization by Smith and Banke (1975) [3] is used:

$$c_D = 10^{-3} \cdot (0.63 + 0.066 \cdot |U_{10}^w|) \tag{2.1.13}$$

This parameterization generates more realistic water levels in periods of high winds compared to the standard parameterization with constant c_D. In equation 2.1.12 the difference between wind speed U_{10}^w and current velocity u is used, which allows to implement this term implicitly by taking u at the new time level.

At the lower boundary bottom friction is implemented in the form

$$\nu \frac{\partial u}{\partial z} = \gamma u \quad \text{and} \quad \nu \frac{\partial v}{\partial z} = \gamma v \quad \text{with} \quad \gamma = \left[\frac{\kappa}{\ln \frac{h}{z_0} - 1} \right]^2 \cdot \sqrt{u^2 + v^2} \tag{2.1.14}$$

This term can also be formulated quasi-implicitly by taking explicit values in the square root and new time level values for u and v in the factors.

The formula for γ results from the assumption of a logarithmic velocity profile and the requirement that the bottom stress must be independent of the thickness of the bottom layer. The logarithmic profile is given by

$$u(z) = \frac{u_*}{\kappa} \ln \frac{z}{z_0} \tag{2.1.15}$$

The mean value of u over the bottom layer is found by integrating 2.1.15 from z_0 to h by making use of $\int \ln x dx = x(\ln x - 1)$:

$$\overline{u} = \frac{u_*}{\kappa} \left[\frac{h}{h - z_0} \ln \frac{h}{z_0} - 1 \right] \tag{2.1.16}$$

which can be approximated further by assuming that h is much bigger than z_0 (in the program it is required to be at least 5x larger):

$$\overline{u} \cong \frac{u_*}{\kappa} \left[\ln \frac{h}{z_0} - 1 \right] \tag{2.1.17}$$

Inserting 2.1.17 into 2.1.14 results in $\tau/\rho = u_*^2$ which is independent of the layer thickness indeed.

2.2 Numerical Formulation

The discrete implementation of equations 2.1.2 proceeds in two steps: First the hydrostatic solution is found by neglecting the non-hydrostatic pressure q. This results in a preliminary velocity field $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$, which are corrected in the second step by the gradient of the non-hydrostatic pressure under the constraint of mass continuity.

2.2.1 Hydrostatic Part

In order to maintain stability of the algorithm with acceptably large time steps the vertical diffusion terms have to be discretized implicitly, and the gradient of the water level is taken semi-implicitly with a factor θ denoting the degree of implicitness with the constraint $\theta \geq 0$. Defining the variables in the grid according to the Arakawa-C notation the discrete equations for the preliminary velocity components are:

$$\begin{aligned} \tilde{u}^{n+1}_{i+\frac{1}{2},j,k} &= Fu^n_{i+\frac{1}{2},j,k} - g\frac{\Delta t}{\Delta x}\left[\theta(\eta^{n+1}_{i+1,j} - \eta^{n+1}_{i,j}) + (1-\theta)(\eta^n_{i+1,j} - \eta^n_{i,j})\right] \\ &+ \Delta t \cdot \frac{\nu_{k+\frac{1}{2}} \cdot \frac{\tilde{u}^{n+1}_{i+\frac{1}{2},j,k+1} - \tilde{u}^{n+1}_{i+\frac{1}{2},j,k}}{\Delta z^n_{i+\frac{1}{2},j,k+\frac{1}{2}}} - \nu_{k-\frac{1}{2}} \cdot \frac{\tilde{u}^{n+1}_{i+\frac{1}{2},j,k} - \tilde{u}^{n+1}_{i+\frac{1}{2},j,k-1}}{\Delta z^n_{i+\frac{1}{2},j,k-\frac{1}{2}}}}{\Delta z^n_{i+\frac{1}{2},j,k}} \end{aligned} \tag{2.2.1}$$

$$\begin{aligned} \tilde{v}^{n+1}_{i,j+\frac{1}{2},k} &= Fv^n_{i,j+\frac{1}{2},k} - g\frac{\Delta t}{\Delta y}\left[\theta(\eta^{n+1}_{i,j+1} - \eta^{n+1}_{i,j}) + (1-\theta)(\eta^n_{i,j+1} - \eta^n_{i,j})\right] \\ &+ \Delta t \cdot \frac{\nu_{k+\frac{1}{2}} \cdot \frac{\tilde{v}^{n+1}_{i,j+\frac{1}{2},k+1} - \tilde{v}^{n+1}_{i,j+\frac{1}{2},k}}{\Delta z^n_{i,j+\frac{1}{2},k+\frac{1}{2}}} - \nu_{k-\frac{1}{2}} \cdot \frac{\tilde{v}^{n+1}_{i,j+\frac{1}{2},k} - \tilde{v}^{n+1}_{i,j+\frac{1}{2},k-1}}{\Delta z^n_{i,j+\frac{1}{2},k-\frac{1}{2}}}}{\Delta z^n_{i,j,k+\frac{1}{2}}} \end{aligned} \tag{2.2.2}$$

$$\begin{aligned} \tilde{w}^{n+1}_{i,j,k+\frac{1}{2}} &= Fw^n_{i,j,k+\frac{1}{2}} \\ &+ \Delta t \cdot \frac{\nu_{k+1} \cdot \frac{\tilde{w}^{n+1}_{i,j,k+\frac{3}{2}} - \tilde{w}^{n+1}_{i,j,k+\frac{1}{2}}}{\Delta z^n_{i,j,k+1}} - \nu_k \cdot \frac{\tilde{w}^{n+1}_{i,j,k+\frac{1}{2}} - \tilde{w}^{n+1}_{i,j,k-\frac{1}{2}}}{\Delta z^n_{i,j,k}}}{\Delta z^n_{i,j,k+\frac{1}{2}}} \end{aligned} \tag{2.2.3}$$

The operator F contains all explicit terms like the Semi-Lagrangian advection, the horizontal diffusion, and the baroclinic pressure gradient. It also contains the Coriolis terms, although they are discretized semi-implicitly. Including these terms into the implicit solution part would destroy the symmetry of the problem deteriorating the convergence properties of the conjugate gradient method. Therefore, the full Coriolis terms are included in F representing some kind of operator split method. The error is likely to be negligible due to the small magnitude of Coriolis terms.

Equation 2.2.3 is only evaluated for non-hydrostatic calculations. Then it serves the purpose of computing a preliminary hydrostatic velocity, which is subsequently corrected by computing the hydrodynamic pressure q (for details see Section 2.2.3). For purely hydrostatic calculations the relation 2.2.5 is used instead.

The hydrostatic water level is found by discretizing equation 2.1.10 with the divergence terms of the preliminary velocity components taken as a weighted mean of the values at old and new time steps:

$$\begin{aligned}
\eta_{i,j}^{n+1} = \eta_{i,j}^{n} & -\frac{\Delta t}{\Delta x}\theta \cdot \left[\sum_{k=m}^{M} \Delta z_{i+\frac{1}{2},j,k}^{n} \tilde{u}_{i+\frac{1}{2},j,k}^{n+1} - \sum_{k=m}^{M} \Delta z_{i-\frac{1}{2},j,k}^{n} \tilde{u}_{i-\frac{1}{2},j,k}^{n+1}\right] \\
& -\frac{\Delta t}{\Delta y}\theta \cdot \left[\sum_{k=m}^{M} \Delta z_{i,j+\frac{1}{2},k}^{n} \tilde{v}_{i,j+\frac{1}{2},k}^{n+1} - \sum_{k=m}^{M} \Delta z_{i,j-\frac{1}{2},k}^{n} \tilde{v}_{i,j-\frac{1}{2},k}^{n+1}\right] \\
& -\frac{\Delta t}{\Delta x}(1-\theta) \cdot \left[\sum_{k=m}^{M} \Delta z_{i+\frac{1}{2},j,k}^{n} u_{i+\frac{1}{2},j,k}^{n} - \sum_{k=m}^{M} \Delta z_{i-\frac{1}{2},j,k}^{n} u_{i-\frac{1}{2},j,k}^{n}\right] \\
& -\frac{\Delta t}{\Delta y}(1-\theta) \cdot \left[\sum_{k=m}^{M} \Delta z_{i,j+\frac{1}{2},k}^{n} v_{i,j+\frac{1}{2},k}^{n} - \sum_{k=m}^{M} \Delta z_{i,j-\frac{1}{2},k}^{n} v_{i,j-\frac{1}{2},k}^{n}\right]
\end{aligned} \tag{2.2.4}$$

Here the limits m and M correspond to the variables **kb** and **kt**, respectively. While m is only a function of space M is a function of both space and time due to the variability of the free surface. Usually $\Delta z_{i+1/2,j,k}^{n}$ is the thickness of the computational cell. In case of partly filled cells like at the bottom or near the surface the wetted height of the cell is taken.

Once the horizontal velocity components are found the vertical velocity component $\tilde{w}$ is updated by applying the continuity constraint equation 2.1.9 to the preliminary velocity components:

$$\begin{aligned}
\tilde{w}_{i,j,k+\frac{1}{2}}^{n+1} = \tilde{w}_{i,j,k-\frac{1}{2}}^{n} & -\frac{\Delta z_{i+\frac{1}{2},j,k}^{n} \tilde{u}_{i+\frac{1}{2},j,k}^{n+1} - \Delta z_{i-\frac{1}{2},j,k}^{n} \tilde{u}_{i-\frac{1}{2},j,k}^{n+1}}{\Delta x} \\
& -\frac{\Delta z_{i,j+\frac{1}{2},k}^{n} \tilde{v}_{i,j+\frac{1}{2},k}^{n+1} - \Delta z_{i,j-\frac{1}{2},k}^{n} \tilde{v}_{i,j-\frac{1}{2},k}^{n+1}}{\Delta y}
\end{aligned} \tag{2.2.5}$$

2.2.2 Matrix Formulation

The discretized equations 2.2.1, 2.2.2, and 2.2.4 can be cast into a more compact form by using matrix notation, where matrices are represented by capital letters:

$$\begin{aligned}
A_{i+\frac{1}{2},j}^{n} U_{i+\frac{1}{2},j}^{n+1} &= G_{i+\frac{1}{2},j}^{n} - g\frac{\Delta t}{\Delta x}\left[\theta(\eta_{i+1,j}^{n+1} - \eta_{i,j}^{n+1})\right] \Delta Z_{i+\frac{1}{2},j}^{n} \\
A_{i,j+\frac{1}{2}}^{n} V_{i,j+\frac{1}{2}}^{n+1} &= G_{i,j+\frac{1}{2}}^{n} - g\frac{\Delta t}{\Delta y}\left[\theta(\eta_{i,j+1}^{n+1} - \eta_{i,j}^{n+1})\right] \Delta Z_{i,j+\frac{1}{2}}^{n} \\
\eta_{i,j}^{n+1} &= \delta_{i,j}^{n} - \theta\frac{\Delta t}{\Delta x}\left[\left(\Delta Z_{i+\frac{1}{2},j}^{n}\right)^{T} U_{i+\frac{1}{2},j}^{n+1} - \left(\Delta Z_{i-\frac{1}{2},j}^{n}\right)^{T} U_{i-\frac{1}{2},j}^{n+1}\right] \\
& \quad - \theta\frac{\Delta t}{\Delta y}\left[\left(\Delta Z_{i,j+\frac{1}{2}}^{n}\right)^{T} V_{i,j+\frac{1}{2}}^{n+1} - \left(\Delta Z_{i,j-\frac{1}{2}}^{n}\right)^{T} V_{i,j-\frac{1}{2}}^{n+1}\right]
\end{aligned} \tag{2.2.6}$$

The vectors U, V, ΔZ, and G are given by

$$U_{i+\frac{1}{2},j}^{n+1} = \begin{pmatrix} \tilde{u}_{i+\frac{1}{2},j,M}^{n+1} \\ \tilde{u}_{i+\frac{1}{2},j,M-1}^{n+1} \\ \tilde{u}_{i+\frac{1}{2},j,M-2}^{n+1} \\ \vdots \\ \tilde{u}_{i+\frac{1}{2},j,m+1}^{n+1} \\ \tilde{u}_{i+\frac{1}{2},j,m}^{n+1} \end{pmatrix}, \quad V_{i,j+\frac{1}{2}}^{n+1} = \begin{pmatrix} \tilde{v}_{i,j+\frac{1}{2},M}^{n+1} \\ \tilde{v}_{i,j+\frac{1}{2},M-1}^{n+1} \\ \tilde{v}_{i,j+\frac{1}{2},M-2}^{n+1} \\ \vdots \\ \tilde{v}_{i,j+\frac{1}{2},m+1}^{n+1} \\ \tilde{v}_{i,j+\frac{1}{2},m}^{n+1} \end{pmatrix}, \quad \Delta Z^{n} = \begin{pmatrix} \Delta z_{M}^{n} \\ \Delta z_{M-1}^{n} \\ \Delta z_{M-2}^{n} \\ \vdots \\ \Delta z_{m+1}^{n} \\ \Delta z_{m}^{n} \end{pmatrix} \tag{2.2.7}$$

$$G^n_{i+\frac{1}{2},j} = \begin{pmatrix} \Delta z^n_M & \cdot\left[Fu^n_{i+\frac{1}{2},j,M} \right. & \left. -g\frac{\Delta t}{\Delta x}(1-\theta)(\eta^n_{i+1,j}-\eta^n_{i,j})\right] + \Delta t\cdot\tau^w_x \\ \Delta z^n_{M-1} & \cdot\left[Fu^n_{i+\frac{1}{2},j,M-1} \right. & \left. -g\frac{\Delta t}{\Delta x}(1-\theta)(\eta^n_{i+1,j}-\eta^n_{i,j})\right] \\ \Delta z^n_{M-2} & \cdot\left[Fu^n_{i+\frac{1}{2},j,M-2} \right. & \left. -g\frac{\Delta t}{\Delta x}(1-\theta)(\eta^n_{i+1,j}-\eta^n_{i,j})\right] \\ & \vdots & \\ \Delta z^n_{m+1} & \cdot\left[Fu^n_{i+\frac{1}{2},j,m+1} \right. & \left. -g\frac{\Delta t}{\Delta x}(1-\theta)(\eta^n_{i+1,j}-\eta^n_{i,j})\right] \\ \Delta z^n_{m} & \cdot\left[Fu^n_{i+\frac{1}{2},j,m} \right. & \left. -g\frac{\Delta t}{\Delta x}(1-\theta)(\eta^n_{i+1,j}-\eta^n_{i,j})\right] \end{pmatrix}, \tag{2.2.8}$$

$$G^n_{i,j+\frac{1}{2}} = \begin{pmatrix} \Delta z^n_M & \cdot\left[Fv^n_{i,j+\frac{1}{2},M} \right. & \left. -g\frac{\Delta t}{\Delta y}(1-\theta)(\eta^n_{i,j+1}-\eta^n_{i,j})\right] + \Delta t\cdot\tau^w_y \\ \Delta z^n_{M-1} & \cdot\left[Fv^n_{i,j+\frac{1}{2},M-1} \right. & \left. -g\frac{\Delta t}{\Delta y}(1-\theta)(\eta^n_{i,j+1}-\eta^n_{i,j})\right] \\ \Delta z^n_{M-2} & \cdot\left[Fv^n_{i,j+\frac{1}{2},M-2} \right. & \left. -g\frac{\Delta t}{\Delta y}(1-\theta)(\eta^n_{i,j+1}-\eta^n_{i,j})\right] \\ & \vdots & \\ \Delta z^n_{m+1} & \cdot\left[Fv^n_{i,j+\frac{1}{2},m+1} \right. & \left. -g\frac{\Delta t}{\Delta y}(1-\theta)(\eta^n_{i,j+1}-\eta^n_{i,j})\right] \\ \Delta z^n_{m} & \cdot\left[Fv^n_{i,j+\frac{1}{2},m} \right. & \left. -g\frac{\Delta t}{\Delta y}(1-\theta)(\eta^n_{i,j+1}-\eta^n_{i,j})\right] \end{pmatrix} \tag{2.2.9}$$

with the following abbreviations for the explicit parts of the wind stress:

$$\tau^w_{x/y} = c_D \cdot \frac{\rho_a}{\rho} \cdot U^w_{x/y,10} \cdot |U^w_{10}| \tag{2.2.10}$$

δ represents the explicit part of equation 2.2.4:

$$\begin{aligned} \delta^n_{i,j} \quad = \quad \eta^n_{i,j} \quad & - \quad (1-\theta)\frac{\Delta t}{\Delta x}\left[\left(\Delta Z^n_{i+\frac{1}{2},j}\right)^T U^n_{i+\frac{1}{2},j} - \left(\Delta Z^n_{i-\frac{1}{2},j}\right)^T U^n_{i-\frac{1}{2},j}\right] \\ & - \quad (1-\theta)\frac{\Delta t}{\Delta y}\left[\left(\Delta Z^n_{i,j+\frac{1}{2}}\right)^T V^n_{i,j+\frac{1}{2}} - \left(\Delta Z^n_{i,j-\frac{1}{2}}\right)^T V^n_{i,j-\frac{1}{2}}\right] \end{aligned} \tag{2.2.11}$$

The matrix A is given by:

$$A = \begin{pmatrix} \Delta z^n_M + a_{M-\frac{1}{2}} + \tau_{12} & -a_{M-\frac{1}{2}} & & 0 \\ -a_{M-\frac{1}{2}} & \Delta z^n_{M-1} + a_{M-\frac{1}{2}} + a_{M-\frac{3}{2}} & -a_{M-\frac{3}{2}} & \\ \vdots & \vdots & \vdots & \vdots \\ 0 & & -a_{m+\frac{1}{2}} & \Delta z^n_m + a_{m+\frac{1}{2}} + \gamma\Delta t \end{pmatrix} \tag{2.2.12}$$

with the abbreviations

$$a_k = \frac{\nu_k \Delta t}{\Delta z_k^n} \quad \text{and} \quad \tau_{12} = \Delta t \cdot c_D \cdot \frac{\rho_a}{\rho} \left| U_{10}^W \right| \tag{2.2.13}$$

Formal substitution of the expressions for $U^{n+1}_{i\pm\frac{1}{2},j}$ and $V^{n+1}_{i,j\pm\frac{1}{2}}$ from 2.2.6 into the equation for $\eta^{n+1}_{i,j}$ of 2.2.6 yields the linear system

$$\begin{aligned}
\eta^{n+1}_{i,j} \quad & -g\frac{\Delta t^2}{\Delta x^2}\theta^2 \cdot \Big[\left[(\Delta Z)^T A^{-1} \Delta Z\right]^n_{i+\frac{1}{2},j} \left(\eta^{n+1}_{i+1,j} - \eta^{n+1}_{i,j}\right) \\
& \qquad - \left[(\Delta Z)^T A^{-1} \Delta Z\right]^n_{i-\frac{1}{2},j} \left(\eta^{n+1}_{i,j} - \eta^{n+1}_{i-1,j}\right) \Big] \\
& -g\frac{\Delta t^2}{\Delta y^2}\theta^2 \cdot \Big[\left[(\Delta Z)^T A^{-1} \Delta Z\right]^n_{i,j+\frac{1}{2}} \left(\eta^{n+1}_{i,j+1} - \eta^{n+1}_{i,j}\right) \\
& \qquad - \left[(\Delta Z)^T A^{-1} \Delta Z\right]^n_{i,j-\frac{1}{2}} \left(\eta^{n+1}_{i,j} - \eta^{n+1}_{i,j-1}\right) \Big] \\
= \delta^n_{i,j} \quad & -\frac{\Delta t}{\Delta x}\theta \cdot \Big[\left[(\Delta Z)^T A^{-1} G\right]^n_{i+\frac{1}{2},j} - \left[(\Delta Z)^T A^{-1} G\right]^n_{i-\frac{1}{2},j} \Big] \\
& -\frac{\Delta t}{\Delta y}\theta \cdot \Big[\left[(\Delta Z)^T A^{-1} G\right]^n_{i,j+\frac{1}{2}} - \left[(\Delta Z)^T A^{-1} G\right]^n_{i,j-\frac{1}{2}} \Big]
\end{aligned} \tag{2.2.14}$$

2.2.3 Non-Hydrostatic Pressure Correction

The non-hydrostatic pressure q is used to correct the preliminary velocity components with the gradient of q such that the continuity constraint for the complete velocity components is satisfied:

$$\begin{aligned}
u^{n+1}_{i+\frac{1}{2},j,k} &= \tilde{u}^{n+1}_{i+\frac{1}{2},j,k} - \frac{\Delta t}{\Delta x} \cdot \left(q^{n+1}_{i+1,j,k} - q^{n+1}_{i,j,k}\right) \\
v^{n+1}_{i,j+\frac{1}{2},k} &= \tilde{v}^{n+1}_{i,j+\frac{1}{2},k} - \frac{\Delta t}{\Delta y} \cdot \left(q^{n+1}_{i,j+1,k} - q^{n+1}_{i,j,k}\right) \\
w^{n+1}_{i,j,k+\frac{1}{2}} &= \tilde{w}^{n+1}_{i,j,k+\frac{1}{2}} - \frac{\Delta t}{\Delta z^{n+1}_{i,j,k+\frac{1}{2}}} \cdot \left(q^{n+1}_{i,j,k+1} - q^{n+1}_{i,j,k}\right)
\end{aligned} \tag{2.2.15}$$

The discretized incompressibility contraint is formulated as:

$$\begin{aligned}
& \frac{\Delta z^{n+1}_{i+\frac{1}{2},j,k} u^{n+1}_{i+\frac{1}{2},j,k} - \Delta z^{n+1}_{i-\frac{1}{2},j,k} u^{n+1}_{i-\frac{1}{2},j,k}}{\Delta x} \\
+ \; & \frac{\Delta z^{n+1}_{i,j+\frac{1}{2},k} v^{n+1}_{i,j+\frac{1}{2},k} - \Delta z^{n+1}_{i,j-\frac{1}{2},k} v^{n+1}_{i,j-\frac{1}{2},k}}{\Delta y} \\
+ \; & w^{n+1}_{i,j,k+\frac{1}{2}} - w^{n}_{i,j,k-\frac{1}{2}} = 0
\end{aligned} \tag{2.2.16}$$

Note that compared to the hydrostatic part of the solution here the updated wet cell thicknesses are used. Inserting the equations for the velocity components 2.2.15 into the continuity equation 2.2.16 leads to a Poisson equation for the dynamic pressure q:

$$
\begin{aligned}
\Delta t \cdot \quad & \left[\frac{\Delta z^{n+1}_{i+\frac{1}{2},j,k}\left(q^{n+1}_{i+1,j,k} - q^{n+1}_{i,j,k}\right) - \Delta z^{n+1}_{i-\frac{1}{2},j,k}\left(q^{n+1}_{i,j,k} - q^{n+1}_{i-1,j,k}\right)}{\Delta x^2} \right. \\
+ \quad & \frac{\Delta z^{n+1}_{i,j+\frac{1}{2},k}\left(q^{n+1}_{i,j+1,k} - q^{n+1}_{i,j,k}\right) - \Delta z^{n+1}_{i,j-\frac{1}{2},k}\left(q^{n+1}_{i,j,k} - q^{n+1}_{i,j-1,k}\right)}{\Delta y^2} \\
+ \quad & \left. \frac{q^{n+1}_{i,j,k+1} - q^{n+1}_{i,j,k}}{\Delta z^{n+1}_{i,j,k+\frac{1}{2}}} - \frac{q^{n+1}_{i,j,k} - q^{n+1}_{i,j,k-1}}{\Delta z^{n+1}_{i,j,k-\frac{1}{2}}} \right] \\
= \quad & \frac{\Delta z^{n+1}_{i+\frac{1}{2},j,k}\tilde{u}^{n+1}_{i+\frac{1}{2},j,k} - \Delta z^{n+1}_{i-\frac{1}{2},j,k}\tilde{u}^{n+1}_{i-\frac{1}{2},j,k}}{\Delta x} \\
+ \quad & \frac{\Delta z^{n+1}_{i,j+\frac{1}{2},k}\tilde{v}^{n+1}_{i,j+\frac{1}{2},k} - \Delta z^{n+1}_{i,j-\frac{1}{2},k}\tilde{v}^{n+1}_{i,j-\frac{1}{2},k}}{\Delta y} \\
+ \quad & \tilde{w}^{n+1}_{i,j,k+\frac{1}{2}} - \tilde{w}^{n}_{i,j,k-\frac{1}{2}}
\end{aligned}
\tag{2.2.17}
$$

2.2.17 constitutes a seven-diagonal linear system for the unknown dynamic pressure q^{n+1} which is symmetric and positive definite. At solid boundaries the flux has to vanish leading to Neumann-type boundaries for q. At the free surface a Dirichlet-type boundary is used setting the hydrodynamic pressure to zero. At open boundaries either the normal velocity component or the dynamic pressure should be specified. Due to lack of better knowledge often the hydrostatic approximation is used at open boundaries (Dirichlet condition $q = 0$).

2.2.4 Tridiagonal Linear Systems

Due to the semi-implicit discretization of diffusion in vertical direction a tridiagonal linear system needs to be solved (see equations 2.2.1 and 2.2.2). Since this type of problem occurs several times within the framework of integrating the discrete equations the general approach for solving this type of linear system is presented here.

The linear system can be written as

$$Ax = f \tag{2.2.18}$$

where the equation for each level $k = M, ..., m$ is defined by

$$a_k \cdot x_{k+1} + b_k \cdot x_k + c_k \cdot x_{k-1} = f_k \tag{2.2.19}$$

Note that the grid point numbering runs from bottom to top (as in the model) in contrast to the mathematical matrix definition where indices run from top rows of the matrix to bottom rows. This renumbering is useful for identifying the loop structure and the variables used in the code with generic variables used in what follows.

The general method to solve system 2.2.18 splits the matrix A into the product of lower and upper triangular matrices L and U, respectively, and solving two subproblems:

$$Ax = f \Longleftrightarrow LUx = f \Longleftrightarrow \left\{ \begin{array}{lcl} Ly & = & f \\ Ux & = & y \end{array} \right. \tag{2.2.20}$$

The lower triangular matrix L has diagonal elements $\tilde{b}_k$ and off-diagonal elements $\tilde{a}_k$, and the upper triangular matrix U has unit diagonal elements and off-diagonal elements $\tilde{c}_k$. The entries of L and U can be found from the following recursive relations:

$$\begin{aligned} \tilde{a}_k &= a_k \\ \tilde{b}_k &= b_k - \tilde{a}_k \cdot \tilde{c}_{k+1} \\ \tilde{c}_k &= c_k/\tilde{b}_k \end{aligned} \tag{2.2.21}$$

The first sub-problem $Ly = f$ is solved by applying

$$y_k = (f_k - \tilde{a}_k \cdot y_{k+1})/\tilde{b}_k \tag{2.2.22}$$

for k running from top to bottom ($k = M, ..., m$). Note that $\tilde{a}_M = 0$. The final solution vector x is found by applying

$$x_k = y_k - \tilde{c}_k \cdot x_{k-1} \tag{2.2.23}$$

for k running from bottom to top ($k = m, ..., M$). Note that $\tilde{c}_m = 0$. Because of the typical two-loop structure in reverse orders this method is sometimes called *double-sweep algorithm*.

2.2.5 Conjugate Gradient Methods

A very efficient method for solving a sparse linear system iteratively is the conjugate gradient method. For the tri-diagonal systems encountered several times in the model a very fast direct method is used (see Section 2.2.4). For the water level η a 5-diagonal system is defined by equation 2.2.14, and for the non-hydrostatic pressure q a 7-diagonal system is given by equation 2.2.17. All these system are symmetric and positive definite by definition of the matrix entries. They are therefore suitable for solving by the conjugate gradient method.

The general outline of the method for solving the system

$$A \cdot e = b \tag{2.2.24}$$

is as follows:

- Guess the initial vector $e^{(0)}$.
- Set the initial direction vector
$$p^{(0)} = r^{(0)} = A \cdot e^{(0)} - b \tag{2.2.25}$$
- Compute the factor α_k
$$\alpha_k = \frac{(r^{(k)}, r^{(k)})}{(p^{(k)}, Ap^{(k)})} \tag{2.2.26}$$
where (a, b) denotes the dot product of a and b.
- Update the solution
$$e^{(k+1)} = e^{(k)} - \alpha_k \cdot p^{(k)} \tag{2.2.27}$$
- Update the residual
$$r^{(k+1)} = r^{(k)} - \alpha_k \cdot Ap^{(k)} \tag{2.2.28}$$
- Compute the factor β_k
$$\beta_k = \frac{(r^{(k+1)}, r^{(k+1)})}{(r^{(k)}, r^{(k)})} \tag{2.2.29}$$
- New direction vector
$$p^{(k+1)} = r^{(k+1)} - \beta_k \cdot p^{(k)} \tag{2.2.30}$$

This procedure is repeated (apart from the initialization steps) until the norm of the residual $(r^{(k)}, r^{(k)})$ is smaller than a prescribed ϵ.

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